CLASSIFICATION OF CONFORMALLY INDECOMPOSABLE INTEGRAL FLOWS ON SIGNED GRAPHS

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Abstract. A conformally indecomposable flow f on a signed graph Σ is a nonzero integral flow that cannot be decomposed into $f = f_1 + f_2$, where f_1, f_2 are nonzero integral flows having the same sign (both ≥ 0 or both ≤ 0) at every edge. This paper is to classify at integer scale conformally indecomposable flows into characteristic vectors of Eulerian cycle-trees — a class of signed graphs having a kind of tree structure in which all cycles can be viewed as vertices of a tree. Moreover, each conformally indecomposable flow other than signed-graphic circuit flows can be further decomposed conformally at half-integer scale into a sum of certain signed-graphic circuit flows. The variety of conformally indecomposable flows of signed graphs is much richer than that of ordinary unsigned graphs.

1. Introduction

A signed graph is an ordinary graph whose each edge is endowed with either a positive sign or a negative sign. The system was formally introduced by Harary [10] who characterized balanced signed graphs up to switching, and was much developed by Zaslavsky [15, 16] who successfully extended most important notions of ordinary graphs to signed graphs, such as circuit, bond, orientation, incidence matrix, Laplacian, and associated matroids, etc. Based on Zaslavsky's work, Chen and Wang [5] introduced flow and tension lattices of signed graphs and obtained fundamental properties on flows and tensions, including a few characterizations of cuts and bonds.

Now it is natural to ask, inside the flow and tension lattices, how integral flows and tensions are built up from more basic integral flows and tensions. More specifically, what does an integral flow or tension look like if it *cannot* be conformally decomposed at integer scale but can be possibly conformally decomposed further at fractional scale? The answer is not only interesting but fundamental in nature because if one considers circuit flows to be at atomic level then conformally indecomposable flows are at molecular level.

For ordinary graphs it is easy to see that conformally indecomposable flows are simply the graphic circuit flows at integer scale. For signed graphs, however, we shall see that conformally indecomposable flows are much richer than that of unsigned graphs because in addition to circuit flows, the fixed spin (signs

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on edges) produces a new class of characteristic vectors of so-called *directed Eulerian cycle-trees*, which are not decomposable conformally at integer scale whereas decomposable conformally (into signed-graphic circuit flows) at half-integer scale. The present paper is to present a complete solution to such a new phenomenon on signed graphs by an algorithmic method. The main result is also obtained by Chen, Wang, and Zaslavsky [7] using a different approach – resolution into a double covering graph.

Let $\Sigma = (V, E, \sigma)$ be a signed graph throughout, where (V, E) is an ordinary finite graph with possible loops and multiple edges, V is the vertex set, E is the edge set, and $\sigma : E \to \{-1, 1\}$ is the sign function. Each edge subset $F \subseteq E$ induces a *signed subgraph* $\Sigma(F) := (V(F), F, \sigma|_F)$, where $V(F)$ is the set of end-vertices of edges in F. A *cycle* of Σ is a simple closed path. The *sign* of a cycle is the product of signs on its edges. A cycle is said to be *balanced (unbalanced)* if its sign is positive (negative). A signed graph is said to be *balanced* if its all cycles are balanced, and *unbalanced* if one of its cycles is unbalanced. A connected component of Σ is called a *balanced (unbalanced) component* if it is balanced (unbalanced) as a signed subgraph.

An *orientation* of a signed graph Σ is an assignment that each edge e is assigned two arrows at its end-vertices u, v as follows: (i) if e is a positive edge, the two arrows are in the same direction; (ii) if e is a negative edge, the two arrows are in opposite directions; see Figure 1. We may think of an arrow on an edge e at its

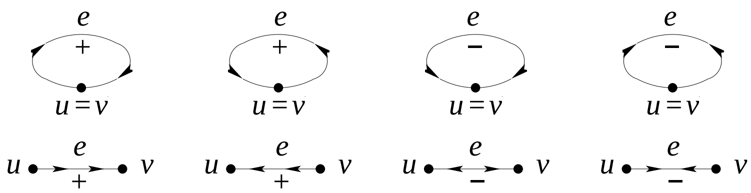

FIGURE 1. Orientations on loops and non-loop edges.

one end-vertex v as $+1$ if the arrow points toward v and -1 if the arrow points away from v. Then there are both $+1$ and -1 for a positive loop at its unique end-vertex, and two $+1$'s or two -1's for a negative loop at its unique end-vertex. So an orientation on Σ can be considered as a multi-valued function $\varepsilon : V \times E \to \{-1, 0, 1\}$ such that

(i) $\varepsilon(v, e)$ has two values $+1$ and -1 if e is a positive loop at its unique end-vertex v, and is single-valued otherwise;
(ii) $\varepsilon(v, e) = 0$ if v is not an end-vertex of the edge e; and
(iii) $\varepsilon(u, e)\varepsilon(v, e) = -\sigma(e)$, $e = uv$.

A signed graph Σ with an orientation ε is called an *oriented signed graph* (Σ, ε). We assume that (Σ, ε) is an oriented signed graph throughout the whole paper.

Let ε_i $(i = 1, 2)$ be orientations on signed subgraphs Σ_i of Σ. The *coupling* of $\varepsilon_1, \varepsilon_2$ is a function $[\varepsilon_1, \varepsilon_2] : E \to \mathbb{Z}$, defined for $e = uv$ by

$$[\varepsilon_1, \varepsilon_2](e) = \begin{cases} 1 & \text{if } e \in \Sigma_1 \cap \Sigma_2,\ \varepsilon_1(v, e) = \varepsilon_2(v, e), \\ -1 & \text{if } e \in \Sigma_1 \cap \Sigma_2,\ \varepsilon_1(v, e) \neq \varepsilon_2(v, e), \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

In other words, $[\varepsilon_1, \varepsilon_2](e) = \varepsilon_1(v, e)\varepsilon_2(v, e)$ if $e = uv$.

Let A be an abelian group and be assumed automatically a $\mathbb{Z}$-module. For each edge e and its end-vertices u, v, let $\mathrm{End}(e)$ denote the multiset $\{u, v\}$. Associated with (Σ, ε) is the *boundary operator* $\partial : A^E \to A^V$ defined by

$$(\partial f)(v) = \sum_{e \in E} \boldsymbol{m}_{v,e} f(e) = \sum_{e \in E,\, w \in \mathrm{End}(e),\, w = v} \varepsilon(w, e) f(e), \tag{1.2}$$

for $f \in A^E$ and $v \in V$, where

$$\boldsymbol{m}_{v,e} = \begin{cases} \varepsilon(v, e) & \text{if } e \text{ is a non-loop,} \\ 2\varepsilon(v, e) & \text{if } e \text{ is a negative loop,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

A function $f : E \to A$ is said to be a *flow* (or *A-flow*) of (Σ, ε) if $(\partial f)(v) = 0$. The set of all A-flows forms an abelian group, called the *flow group* of (Σ, ε) with values in A, denoted $F(\Sigma, \varepsilon; A)$. We call $F(\Sigma, \varepsilon) := F(\Sigma, \varepsilon; \mathbb{R})$ the *flow space*, and $Z(\Sigma, \varepsilon) := F(\Sigma, \varepsilon; \mathbb{Z})$ the *flow lattice* of (Σ, ε). The *support* of f is the edge subset

$$\mathrm{supp}\, f = \{e \in E \mid f(e) \neq 0\}. \tag{1.4}$$

For further information about flows of signed graphs, see [1, 4, 5, 6, 11]. For notions of ordinary graphs, we refer to the books [2, 3, 9].

A flow is said to be *nonzero* if its support is nonempty. A nonzero integral flow f is said to be *conformally decomposable* if f can be written as

$$f = f_1 + f_2,$$

where f_1, f_2 are nonzero integral flows having the same sign (both nonnegative or both nonpositive) at every edge, that is, $f_1(e) f_2(e) \geq 0$ for all $e \in E$. Nonzero integral flows that are not conformally decomposable are said to be *conformally indecomposable*. A nonzero integral flow f is said to be *elementary* if it is conformally indecomposable and there is no nonzero integral flow g such that $\mathrm{supp}\, g$ is properly contained in $\mathrm{supp}\, f$. Compare with Tutte's definition of elementary chains [13, 14].

Let W be a walk of length n in Σ and be written as a vertex-edge sequence

$$W = u_0 x_1 u_1 x_2 \ldots u_{n-1} x_n u_n, \tag{1.5}$$

where each x_i is an edge with end-vertices u_{i-1}, u_i. The walk W is said to be *closed* if the *initial vertex* u_0 is the same as the *terminal vertex* u_n. The *sign* of W is the product

$$\sigma(W) = \prod_{i=1}^{n} \sigma(x_i). \tag{1.6}$$

The *support* of W is the set $\mathrm{supp}\, W$ of edges x_i $(i = 1, \ldots, n)$ without repetition. We may think of W as a multiset

$$M(W) = \{x_1, x_2, \ldots, x_n\} \tag{1.7}$$

(with repetition allowed) of n edges on $\mathrm{supp}\, W$.

A *direction* of W is a function ε_W with values either 1 or -1, defined for all vertex-edge pairs (u_{i-1}, x_i) and (u_i, x_i), such that

$$\varepsilon_W(u_{i-1}, x_i) \varepsilon_W(u_i, x_i) = -\sigma(x_i),$$

$$\varepsilon_W(u_i, x_i) + \varepsilon_W(u_i, x_{i+1}) = 0.$$

It follows that for each direction ε_W,

$$\varepsilon_W(u_n, x_n) = -\sigma(W) \varepsilon_W(u_0, x_1). \tag{1.8}$$

Every walk has exactly two opposite directions. A walk W with a direction ε_W is called a *directed walk*, denoted (W, ε_W). If W is closed, its direction ε_W satisfies

$$\varepsilon_W(u_0, x_1) + \varepsilon_W(u_n, x_n) = 0$$

at the initial and terminal vertex $u_0 (= u_n)$ if and only if W has positive sign.

A directed walk (W, ε_W) is said to be *midway-back avoided*, provided that if $u_\alpha = u_\beta$ with $0 < \alpha < \beta < n$ in (1.5) then

$$\varepsilon_W(u_\beta, x_\beta) = \varepsilon_W(u_\alpha, x_{\alpha+1}). \tag{1.9}$$

Figure 2 demonstrates four possible orientation patterns at a double vertex in a directed midway-back avoided walk.

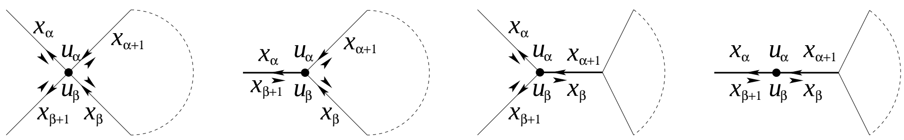

FIGURE 2. Orientation patterns at a double vertex.

An *Eulerian walk* is a closed positive walk whose direction has the same orientation on repeated edges of each fixed edge. We shall see that a midway-back avoided closed positive walk is necessarily a directed Eulerian walk and has no triple vertices. An Eulerian walk with a direction is called a *directed Eulerian walk*.

An Eulerian walk W is said to be *minimal* if there is no Eulerian walk W' such that W' is properly contained in W as edge multisets, and said to be *elementary* if it is minimal and there is no Eulerian walk W' such that $\operatorname{supp} W'$ is properly contained in $\operatorname{supp} W$ as edge subsets. A minimal Eulerian walk with a direction is called a *minimal directed Eulerian walk*.

Let (W, ε_W) be a directed closed positive walk having W given by (1.5). The *characteristic vector* of (W, ε_W) on (Σ, ε) is a function $f_{(W, \varepsilon_W)} : E \to \mathbb{Z}$ defined by

$$f_{(W, \varepsilon_W)}(x) = \sum_{x_i \in W,\, x_i = x} [\varepsilon, \varepsilon_W](x_i). \tag{1.10}$$

By Lemma 2.1, $f_{(W, \varepsilon_W)}$ is an integral flow on (Σ, ε). Whenever $\varepsilon_W = \varepsilon$ on W, we simply write $f_{(W, \varepsilon_W)}$ as f_W.

Given a real-valued function f on E. Let ε_f be the orientation on Σ defined by

$$\varepsilon_f(u, x) = \begin{cases} -\varepsilon(u, x) & \text{if } f(e) < 0,\ x = uv, \\ \varepsilon(u, x) & \text{otherwise.} \end{cases} \tag{1.11}$$

It is trivial that f is a flow on (Σ, ε) if and only if the absolute value function $|f|$ is a flow on (Σ, ε_f). Moreover, $|f| = [\varepsilon, \varepsilon_f] \cdot f$.

A *cycle-tree* of Σ is a connected signed subgraph T which can be decomposed into edge-disjoint cycles C_i (called *block cycles*) and vertex-disjoint simple paths P_j (called *block paths*), denoted $T = \{C_i, P_j\}$, satisfying the four conditions:

(i) $\{C_i\}$ is the collection of all cycles (simple closed paths) in T.
(ii) The intersection of two cycles is either empty or a single vertex (called an *intersection vertex*).
(iii) Each P_j intersects exactly two cycles and the intersections are exactly the initial and terminal vertices of P_j (also called *intersection vertices*).

(iv) Each intersection vertex is a *cut-point* (a vertex whose removal increases the number of connected components of the underlying graph as a topological space of 1-dimensional CW complex), also known as a *separating vertex* [3, p.119].

A cycle-tree is said to be *Eulerian* if it further satisfies

(v) *Parity Condition*: Each balanced cycle has even number of intersection vertices, while each unbalanced cycle has odd number of intersection vertices.

We call a block cycle in a cycle-tree to be an *end-block cycle* if it has exactly one intersection vertex. The name cycle-tree is justified as follows: if one converts each block cycle C_i into a vertex, each common intersection vertex of two block cycles into an edge adjacent with the two vertices converted from the two block cycles, and keep each block path P_j connecting two vertices converted from the two block cycles connected by P_j, then the graph so obtained is indeed a tree.

An orientation ε_T on a cycle-tree T is called a *direction* if (T, ε_T) has neither sink nor source, and for each block cycle C, the restriction (C, ε_T) has either a sink or a source at each cut-point of T on C. We shall see that T admits a direction if and only if T satisfies the Parity Condition, and the direction is unique up to opposite sign. An Eulerian cycle-tree T with a direction ε_T is called a *directed Eulerian cycle-tree* (T, ε_T). For instance, the oriented signed graph given in Figure 3 is an Eulerian cycle-tree with a direction.

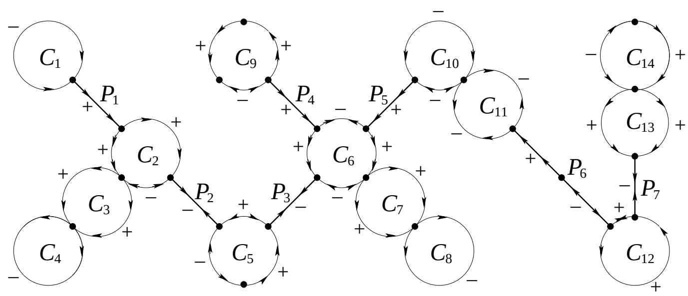

FIGURE 3. An Eulerian cycle-tree and its direction.

Let $T = \{C_i, P_j\}$ be an Eulerian cycle-tree of Σ. The *indicator* of T, denoted $I_{\Sigma(T)}$, is a function $I_T : E \to \mathbb{Z}$ defined by

$$I_T(e) = \begin{cases} 1 & \text{if } e \text{ belongs to block cycles,} \\ 2 & \text{if } e \text{ belongs to block paths,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.12}$$

Given a direction ε_T of T. Viewing both (Σ, ε) and (T, ε_T) as oriented signed subgraphs of Σ, we have the coupling $[\varepsilon, \varepsilon_T]$. The product function $[\varepsilon, \varepsilon_T] \cdot I_T$ determines a vector in $\mathbb{Z}^E$ and is an integral flow of (Σ, ε) by Theorem 3.4, called the *characteristic vector* of the directed Eulerian cycle-tree (T, ε_T) for (Σ, ε).

An Eulerian cycle-tree is called a *(signed-graphic) circuit* if it does not contain properly any Eulerian cycle-tree. We shall see that each circuit C must be one of the following three types.

- *Type I*: C consists of a single balanced cycle.

- *Type II*: C consists of two edge-disjoint unbalanced cycles C_1, C_2 and the intersection $C_1 \cap C_2$ is a single vertex, written $C = C_1C_2$.
- *Type III*: C consists of two vertex-disjoint unbalanced cycles C_1, C_2, and a simple path P of positive length, such that $C_1 \cap P$ is the initial vertex and $C_2 \cap P$ the terminal vertex of P, written $C = C_1PC_2$.

The present definition of circuit looks different from that defined by Zaslavsky [15] and that adopted in [5, 6], but they are equivalent. The following characterization of signed-graphic circuits shows the motivation of the concept.

Characterization of Signed-Graphic Circuits. *Let f be a nonzero integral flow of (Σ, ε). Then the following statements are equivalent.*

(a) *f is elementary.*
(b) *f is the characteristic vector of a directed circuit.*
(c) *There exists an elementary directed Eulerian walk (W, ε_W) such that*

$$f = f_{(W,\varepsilon_W)}.$$

Remark. The characterization of signed-graphic circuits was obtained by Bouchet [4, p. 283] (Corollary 2.3), using Zaslavsky's definition of circuits [15]. As Zaslavsky pointed out himself, the central observation of [15, p. 53] is the existence of a matroid over the edge set of a signed graph whose circuits are exactly those of Types I, II, III. Bouchet [4] assumed (without argument) that Zaslavsky's matroid is the same as the matroid whose circuits are the supports of elementary flows.

Indeed, it is trivial to see that the circuits of the former are the circuits of the latter. However, the converse is not so obvious that it needs no argument, although it is anticipated. Corollary 3.6 implies that the converse is indeed true. Now it is logically clear and aesthetically complete that the matroid constructed by Zaslavsky [15] for a signed graph is the same matroid whose circuits are the supports of elementary chains (= elementary flows) of the signed graph in the sense of Tutte [14]; so are their dual matroids.

Classification of Conformally Indecomposable Integral Flows. *Let f be a nonzero integral flow of an oriented signed graph (Σ, ε).*

(a) Then f is conformally indecomposable if and only if $\operatorname{supp} f$ *is an Eulerian cycle-tree T and*

$$f = [\varepsilon, \varepsilon_f] \cdot I_T.$$

(b) If T is an Eulerian cycle-tree other than a circuit, then for each closed walk W of minimum length that uses all edges of T, there is a decomposition

$$W = C_0P_1C_1 \ldots P_kC_kP_{k+1}, \quad k \geq 1,$$

where C_i are entire end-block cycles of T and $C_iP_{i+1}C_{i+1}$ are circuits of Type III with $C_{k+1} = C_0$, such that

$$I_T = \frac{1}{2}\sum_{i=0}^{k} I_{\Sigma(C_iP_{i+1}C_{i+1})}.$$

Here the half-integer phenomenon is similar to Corollary 1.4 of Geelen and Guenin [8, p. 283] in spirit.

2. Flow Reduction Algorithm

It is easy to see that the conformal decomposability of an integral flow f on (Σ,ε) is equivalent to the conformal decomposability of the flow $|f|$ on (Σ,ε_f); see Lemma 8 of [7]. So without loss of generality, to decompose an integral flow, one only needs to consider nonnegative nonzero integral flows of (Σ,ε). The following *Flow Reduction Algorithm* (*FRA*) finds explicitly a minimal directed Eulerian walk from a given nonzero integral flow. To be self-contained, let us first state the following lemma, saying that the characteristic vector of a directed closed positive walk is an integral flow.

Lemma 2.1. *Let* (W,ε_W) *be a directed closed walk. Then the function* $f_{(W,\varepsilon_W)}$ *defined by* (1.10) *is an integral flow of* (Σ,ε).

Proof. See Lemma 4.2 of [5, p. 273] and Lemma 3 of [7]. $\square$

Flow Reduction Algorithm (FRA). Given a nonzero integral flow f on (Σ,ε).

Step 0. *Choose an edge* x_1 *in* $\operatorname{supp} f$ *with end-vertices* u_0, u_1. *Initiate a half-closed and half-open walk* u_0x_1. *Set* $W := u_0x_1$ *and* $\ell := 1$. *Go to* Step 1.

Step 1. *If* $u_\ell \notin W$, *go to* Step 2. *If* $u_\ell \in W$, *say,* $u_\ell = u_\beta$ *with the greatest index* $\beta < \ell$, *go to* Step 3.

Step 2. *There exists an edge* $x_{\ell+1}$ *in* $\operatorname{supp} f'$ *other than* x_ℓ, *where* $f' := f - f_{(W,\varepsilon_f)}$, *having end-vertices* $u_\ell, u_{\ell+1}$ *such that* $\varepsilon_f(u_\ell, x_{\ell+1}) = -\varepsilon_f(u_\ell, x_\ell)$. *Set* $W := Wu_\ell x_{\ell+1}$ *and* $\ell := \ell + 1$. *Return to* `STEP 1`.

Step 3. *If* u_β *repeats a vertex in* W *at time* β, *say,* $u_\alpha = u_\beta$ *with* $\alpha < \beta < \ell$, `STOP`. *For the case* $\varepsilon_f(u_\ell, x_\ell) = -\varepsilon_f(u_\beta, x_{\beta+1})$, *set*

$$W := u_\beta x_{\beta+1} u_{\beta+1} \dots u_{\ell-1} x_\ell u_\ell. \tag{2.1}$$

For the case $\varepsilon_f(u_\ell, x_\ell) = \varepsilon_f(u_\beta, x_{\beta+1})$, *set*

$$W := u_\alpha x_{\alpha+1} u_{\alpha+1} \dots u_{\ell-1} x_\ell u_\ell; \tag{2.2}$$

see Figure 4(a). *Then* (W,ε_f) *is a* directed Eulerian walk. *If* u_β *does not repeat any vertex in* W *before time* β, *go to* Step 4.

Step 4. *If there exist repeated vertices* u_α, u_γ *in* W *with* $\alpha < \beta < \gamma$ *such that* $u_\alpha = u_\gamma$, `Stop`. *For the case* $\varepsilon_f(u_\ell, x_\ell) = -\varepsilon_f(u_\beta, x_{\beta+1})$, *set* W *to be of (2.1). For the case* $\varepsilon_f(u_\ell, x_\ell) = \varepsilon_f(u_\beta, x_{\beta+1})$, *set*

$$W := u_\beta x_\beta u_{\beta-1} \dots u_{\alpha+1} x_{\alpha+1} u_\alpha (u_\gamma) x_{\gamma+1} u_{\gamma+1} \dots u_{\ell-1} x_\ell u_\ell; \tag{2.3}$$

see Figure 4(b). *Then* (W,ε_f) *is a* directed Eulerian walk. *Otherwise, go to* Step 5.

Step 5. *If* $\varepsilon_f(u_\ell, x_\ell) = -\varepsilon_f(u_\beta, x_{\beta+1})$, stop. *Set* W *to be of (2.1). Then* (W,ε_f) *is a* directed Eulerian walk. *If* $\varepsilon_f(u_\ell, x_\ell) = \varepsilon_f(u_\beta, x_{\beta+1})$, *return to* Step 2.

Let W be a dynamic walk obtained by FRA. It is clear from Step 3 that the multiplicity of each vertex of W is at most two. So W has only possible double vertices and possible double edges. At each double vertex of W, say $u_\alpha = u_\beta$ with $\alpha < \beta$, Step 5 implies that

$$\varepsilon(u_\beta, x_\beta) = \varepsilon_W(u_\alpha, x_{\alpha+1}) = -\varepsilon_W(u_\alpha, x_\alpha) = -\varepsilon_W(u_\beta, x_{\beta+1}). \tag{2.4}$$

This means that (W,ε_W) is a directed Eulerian walk when FRA stops. It is possible that $(u_\beta, x_{\beta+1}) = (u_\alpha, x_\alpha)$; if so, the repeated edges $x_\alpha, x_{\beta+1}$ have the same

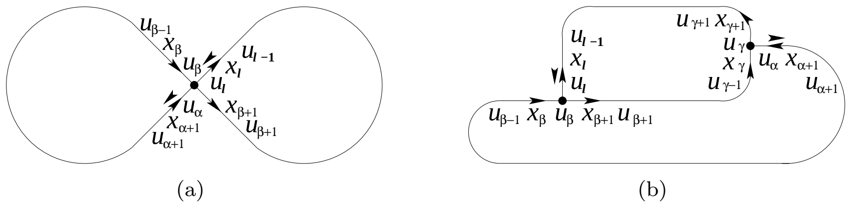

FIGURE 4. Two directed Eulerian walks found by FRA

orientation in (W, ε_W). See Figure 2 for four possible patterns. STEP 4 implies that each double vertex of W must be a cut-point of $\Sigma(W)$.

In STEP 2, either $u_\ell \notin W$ or $u_\ell = u_\beta$ with $\varepsilon_f(u_\ell, x_\ell) = \varepsilon_f(u_\beta, x_{\beta+1})$, both functions f_W, f' are not flows. In fact, $\partial f_{(W,\varepsilon_f)}(u_\ell) = \varepsilon_f(u_\ell, x_\ell) \neq 0$ and $\partial f'(u_\ell) = -\varepsilon_f(u_\ell, x_\ell) \neq 0$. This means that there exists an edge $x_{\ell+1}$ in $\operatorname{supp} f'$ at u_ℓ such that $\varepsilon_f(u_\ell, x_{\ell+1}) = -\varepsilon_f(u_\ell, x_\ell)$. Then the length of W increases one and the cardinality of the multiset $(E, |f|)$ decreases one. Continue this procedure, FRA stops with a directed closed walk (W, ε_f), for $|f|$ is finite.

Lemma 2.2. *Let W be a directed walk. Then FRA finds no directed closed positive walk along W if and only if FRA finds no directed closed positive walk along W^{-1}.*

Proof. It seems to be quite obvious. In fact, let FRA find a directed closed positive walk along W. Then W contains one of the three patterns of closed walks in Figure 5 with $\alpha < \beta < \gamma < \delta$:

(a) $\varepsilon_W(u_\beta, x_\beta) = -\varepsilon_W(u_\alpha, x_{\alpha+1})$;
(b) $\varepsilon_W(u_\beta, x_\beta) = \varepsilon_W(u_\alpha, x_{\alpha+1})$, $\varepsilon_W(u_\gamma, x_\gamma) = -\varepsilon_W(u_\alpha, x_{\alpha+1})$;
(c) $\varepsilon_W(u_\gamma, x_\gamma) = \varepsilon_W(u_\alpha, x_{\alpha+1})$, $\varepsilon_W(u_\delta, x_\delta) = \varepsilon_W(u_\beta, x_{\beta+1})$.

The reversals of patterns (a), (b) and (c), as subwalks in W^{-1}, have the same patterns as (a), (b) and (c) respectively. The subwalks from u_α to u_β in (a), (b), (c) may contain some double vertices and double edges; so do the subwalks from u_β to u_γ in (b) and (c); and so does the subwalk from u_γ to u_δ in (c).

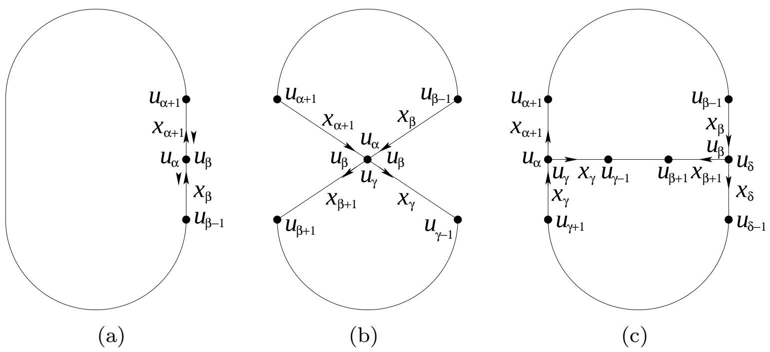

FIGURE 5. Three patterns that FRA stops.

Note that when FRA is applied to W, the algorithm may stop and find a directed closed positive walk before it reaches u_β in (a), or before it reaches u_γ in (b), or before it reaches u_δ in (c). If so, when FRA is applied to W^{-1}, the algorithm stops and finds a directed closed positive walk along W^{-1} before it reaches $u_{\delta^{-1}}$, or $u_{\gamma^{-1}}$, or $u_{\beta^{-1}}$, or $u_{\alpha^{-1}}$. If not, when FRA is applied to W^{-1}, the algorithm stops and finds a directed closed positive walk when it reaches $u_{\alpha^{-1}}$. This means that FRA finds a directed closed positive walk along W^{-1}.

Conversely, let FRA find a directed closed positive walk along W^{-1}. Then FRA finds a directed closed positive walk along the walk $(W^{-1})^{-1}$, which is W. □

Lemma 2.3. *Let (W, ε_W) be a directed midway-back avoided walk. Then*

(a) *The walk W has only possible double vertices, that is, the multiplicity of each vertex and of each edge in W is at most two.*
(b) *The direction ε_W has the same orientation on repeated edges of W.*
(c) *If W is a closed positive walk, then (W, ε_W) is a directed Eulerian walk.*

Proof. Write $W = u_0 x_1 u_1 x_2 \ldots u_{n-1} x_n u_n$.

(a) Suppose there is a vertex appeared three times in W, say, $u_\alpha = u_\beta = u_\gamma$ with $\alpha < \beta < \gamma$; see Figure 6. Since (W, ε_W) is midway-back avoided, we have

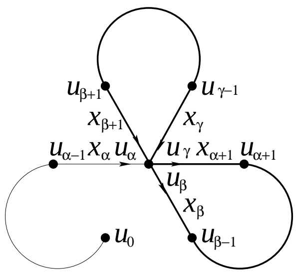

FIGURE 6. The pattern of a triple point.

$$\varepsilon_W(u_\beta, x_\beta) = \varepsilon_W(u_\alpha, x_{\alpha+1}),$$
$$\varepsilon_W(u_\gamma, x_\gamma) = \varepsilon_W(u_\beta, x_{\beta+1}),$$
$$\varepsilon_W(u_\gamma, x_\gamma) = \varepsilon_W(u_\alpha, x_{\alpha+1}).$$

Then

$$\varepsilon_W(u_\gamma, x_\gamma) = -\varepsilon_W(u_\beta, x_\beta) = -\varepsilon_W(u_\alpha, x_{\alpha+1}) = -\varepsilon_W(u_\gamma, x_\gamma),$$

which is a contradiction.

(b) Let $u_\alpha = u_\beta$ with $\alpha < \beta$ and let $x_{\beta+1}$ $(= x_\alpha)$ be a repeated edge. Then $u_{\beta+1} = u_{\alpha-1}$. Suppose $\varepsilon_W(u_\beta, x_{\beta+1}) = -\varepsilon_W(u_\alpha, x_\alpha)$. Then

$$\varepsilon_W(u_{\beta+1}, x_{\beta+1}) = -\varepsilon_W(u_{\alpha-1}, x_\alpha).$$

This means that (W, ε_W) is midway-back at $u_{\alpha-1}$, which is a contradiction. So $\varepsilon_W(u_\beta, x_{\beta+1}) = \varepsilon_W(u_\alpha, x_\alpha)$. This means that ε_W has the same orientation on repeated edges of W.

(c) It follows from (1.8) that $\varepsilon_W(u_n, x_n) = -\varepsilon_W(u_0, x_1)$. Hence (W, ε_W) is a directed Eulerian walk by (b). □

Lemma 2.4. *Let (W, ε_f) be a directed closed positive walk found by FRA. Then*

(a) *(W, ε_f) is a midway-back avoided walk.*
(b) *Each double vertex in W is a cut-point of $\Sigma(W)$.*

Proof. (a) The directed walk (W, ε_f) satisfies (2.4). By definition (W, ε_f) is midway-back avoided.

(b) Assume that FRA stops at time ℓ and finds a directed closed walk (W, ε_f), but did not stop before ℓ. The format of W in the forms (2.1) and (2.2) have the same pattern of increasing indices. However, the format of W in the form (2.3) is special; its indices from u_β to u_α decrease. We may reduce the format of W in the form (2.3) to the form whose indices increase as follows.

Consider the directed walk (W', ε_f), where $W' = W_1W_2$,

$$W_1 = u_\alpha x_{\alpha+1}u_{\alpha+1}\dots u_\beta x_{\beta+1}u_{\beta+1}\dots u_{\gamma-1}x_\gamma u_\gamma,$$
$$W_2 = u_\gamma x_{\gamma+1}u_{\gamma+1}\dots u_{\ell-1}x_\ell u_\ell, \quad u_\gamma = u_\alpha,\ u_\ell = u_\beta.$$

Applying FRA to W_1W_2, the algorithm cannot stop before time ℓ, but stops at time ℓ and finds the directed closed positive walk W. Of course, FRA finds no directed closed positive walk along W_1. Writing W_1^{-1} in increasing-order of indices and applying FRA to W_1^{-1}, by Lemma 2.2 the algorithm finds no directed closed walk along W_1^{-1}. Now applying FRA to $W_1^{-1}W_2$, the algorithm cannot stop before time ℓ, but stops at time ℓ and finds the same directed closed walk W, having indices in increasing-order.

Without loss of generality we may assume that (W, ε_f) (obtained by FRA) has the form

$$W = u_0x_1u_1x_2\dots u_{\ell-1}x_\ell u_\ell, \quad u_0 = u_\ell. \tag{2.5}$$

Suppose W has a double vertex u that is not a cut-point of $\Sigma(W)$, say, $u = u_\delta = u_\eta$ with $\delta < \eta$. Remove the vertex u from $\Sigma(W)$. Since u is a double vertex, $\Sigma(W) \smallsetminus \{u\}$ is the union of two open walks

$$x_{\delta+1}u_{\delta+1}\dots u_{\eta-1}x_\eta, \quad x_{\eta+1}u_{\eta+1}\dots u_{\ell-1}x_\ell u_\ell(u_0)x_1u_1\dots u_{\delta-1}x_\delta.$$

Since u is not a cut-point, the two open walks must intersect at a vertex, say,

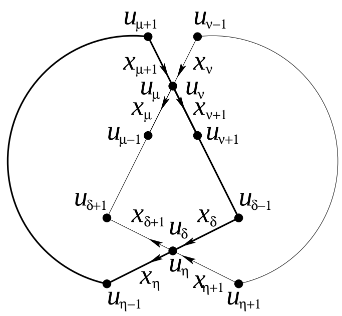

FIGURE 7. A double vertex that is not a cut-point.

$u_\mu = u_\nu$, where $\delta < \mu < \eta$ and either $\eta < \nu$ or $\nu < \delta$. With the indices modulo ℓ, the closed walk W can be written as the form (see Figure 7)

$$W = u_\delta x_{\delta+1}u_{\delta+1}\dots x_\mu u_\mu x_{\mu+1}\dots x_\eta u_\eta x_{\eta+1}\dots x_\nu u_\nu x_{\nu+1}\dots u_{\delta-1}x_\delta u_\delta.$$

Consider the case $\delta < \mu < \eta < \nu$. If $\nu < \ell$, FRA stops in STEP 4 at time ν and finds the directed closed positive walk

$$u_\mu x_\mu u_{\mu-1}\dots u_{\delta+1}x_{\delta+1}u_\delta(u_\eta)x_{\eta+1}u_{\eta+1}\dots u_{\nu-1}x_\nu u_\nu$$

in Figure 7; this is a contradiction. If $\nu = \ell$, then $u_\nu x_{\nu+1} u_{\nu+1} = u_0 x_1 u_1$, FRA stops in Step 4 at time η and finds the directed closed positive walk

$$u_\nu x_{\nu+1} u_{\nu+1} \ldots u_{\delta-1} x_\delta u_\delta (u_\eta) x_\eta u_{\eta-1} \ldots u_{\mu+1} x_{\mu+1} u_\mu$$

in Figure 7; this is a contradiction. For the case $\nu < \delta < \mu < \eta$, it is analogous to the case $\delta < \mu < \eta < \nu$. □

Theorem 2.5. *Let (W, ε_W) be a directed closed positive walk such that*

(i) *(W, ε_W) is a directed midway-back avoided walk;*
(ii) *each double vertex in W is a cut-point of $\Sigma(W)$.*

Then $\Sigma(W)$ is an Eulerian cycle-tree, the restriction of ε_W to $\Sigma(W)$ is a direction on the cycle-tree, and W uses each edge of block cycles once and each edge of block paths twice, crossing from one block to the other block at each cut-vertex.

Proof. Lemma 2.3 implies that W has only possible double vertices and possible double edges. Since every double vertex of W is a cut-point of $\Sigma(W)$, then the connected components of the signed subgraph induced by double edges of W are simple paths (of possible zero length), called *double-edge paths* of $\Sigma(W)$. Remove the internal part of each double-edge path of positive length from $\Sigma(W)$, we obtain an Eulerian graph whose vertex degrees are either 2 or 4. The Eulerian graph can be decomposed into edge-disjoint cycles, called *block cycles*. Each double-edge path (of possible zero length) connects exactly two block cycles. Since $\Sigma(W)$ is connected and each double vertex of W is a cut-point of $\Sigma(W)$, it follows that $\Sigma(W)$ is a cycle-tree.

It is clear that W uses each edge of block cycles once and each edge of block paths twice, and crosses from one block to the other block at each cut-vertex. Since (W, ε_W) is midway-back avoided, Lemma 2.3(b) implies that ε_W has the same orientation on repeated edges of W. So ε_W is a direction on the cycle-tree $\Sigma(W)$. Now Theorem 3.2 implies that $\Sigma(W)$ satisfies the Parity Condition. Hence $\Sigma(W)$ is an Eulerian cycle-tree. □

Corollary 2.6. *Let (W, ε_f) be a directed closed walk found by FRA. Then $\Sigma(W)$ is an Eulerian cycle-tree with direction ε_f, and W uses each edge of block cycles once and each edge of block paths twice, crossing from one block to the other block at each cut-vertex.*

Proof. It follows from Lemma 2.4 and Theorem 2.5. □

Theorem 2.7 (Flow Reduction Theorem)**.** *Let f be a nonzero integral flow of (Σ, ε). Then there exist minimal directed Eulerian walks (W_i, ε_f) and Eulerian cycle-trees $T_i = \Sigma(W_i)$ such that f can be conformally decomposed into*

$$f = \sum f_{(W_i, \varepsilon_f)} = \sum [\varepsilon, \varepsilon_f] \cdot I_{T_i}, \tag{2.6}$$

where $f_{(W_i, \varepsilon_f)}$ are given by (1.10) and I_{T_i} by (1.12).

Furthermore, if f is conformally indecomposable, then there exists a minimal directed Eulerian walk (W, ε_f) and an Eulerian cycle-tree $T = \Sigma(W)$ such that

$$f = f_{(W, \varepsilon_f)} = [\varepsilon, \varepsilon_f] \cdot I_T. \tag{2.7}$$

Proof. Consider the nonnegative integral flow $|f|$ of (Σ, ε_f), where ε_f is defined by (1.11). Let $\Sigma(f)$ denote the signed subgraph induced by the edge subset $\operatorname{supp} f$. Then FRA finds a directed Eulerian walk (W_1, ε_f) on the oriented signed graph

$(\Sigma(f), \varepsilon_f)$, such that $f_{W_1} \leq |f|$, where f_{W_1} is given by (1.10) with $\varepsilon = \varepsilon_f$. Corollary 2.6 implies that $\Sigma(W_1)$ is an Eulerian cycle-tree T_1, and Theorem 3.5 implies that W_1 is a minimal Eulerian walk. Then Theorem 3.4 implies $f_{W_1} = I_{T_1}$, the indicator function of T_1 defined by (1.12).

If $f_1 := |f| - f_{W_1} \neq 0$, then FRA finds a minimal directed Eulerian walk (W_2, ε_f) on $(\Sigma(f_1), \varepsilon_f)$, such that $f_{W_2} \leq |f| - f_{W_1}$ and $f_{W_2} = I_{T_2}$, where T_2 is the Eulerian cycle-tree $\Sigma(W_2)$. Likewise, if $f_2 := |f| - f_{W_1} - f_{W_2} \neq 0$, then FRA finds a minimal directed Eulerian walk (W_3, ε_f) on $(\Sigma(f_2), \varepsilon_f)$, such that $f_{W_3} \leq |f| - f_{W_1} - f_{W_2}$ and $f_{W_3} = I_{T_3}$, where T_3 is the Eulerian cycle-tree $\Sigma(W_3)$. Continue this procedure, we obtain minimal directed Eulerian walks

$$(W_1, \varepsilon_f),\ (W_2, \varepsilon_f),\ \ldots,\ (W_k, \varepsilon_f)$$

on $(\Sigma(f), \varepsilon_f)$, such that $|f| = \sum_{i=1}^{k} f_{W_i} = \sum_{i=1}^{k} I_{T_i}$, where T_i are the Eulerian cycle-trees $\Sigma(W_i)$ and $f_{W_i} = I_{T_i}$. Note that

$$f = [\varepsilon, \varepsilon_f] \cdot |f|, \quad f_{(W_i, \varepsilon_f)} = [\varepsilon, \varepsilon_f] \cdot f_{W_i}.$$

We obtain $f = \sum_{i=1}^{k} f_{(W_i, \varepsilon_f)} = \sum_{i=1}^{k} [\varepsilon, \varepsilon_f] \cdot I_{T_i}$.

If f is conformally indecomposable, by definition we must have $k = 1$. □

3. Characterizations of Eulerian Cycle-trees

This section is to establish properties satisfied by Eulerian cycle-trees such as the Existence and Uniqueness of Direction, the Minimality, and the Half-Integer Scale Decomposition. These results are interesting and important for their own right; some of them have been used in Section 2. Furthermore, we shall see the equivalence of conformally indecomposable flows, minimal Eulerian walks, and Eulerian cycle-trees. The byproduct is the equivalence of circuits, elementary flows, and elementary Eulerian walks, and the classification of circuits.

Let $T = \{C_i, P_j\}$ be a cycle-tree with block cycles C_i and block paths P_j throughout. We choose a block cycle C_0 and write it as a closed walk

$$W_0 = u_0 x_1 u_1 x_2 \ldots u_{l-1} x_l u_l, \quad u_l = u_0. \tag{3.1}$$

If T has two or more block cycles, we require C_0 to be an end-block cycle, having u_0 as its unique intersection vertex. Let P be the block path (of possible zero length) from the vertex u_0 on C_0 to a vertex w_0 on another block cycle C_1. We write

$$P = v_0 y_1 v_1 y_2 \ldots v_{m-1} y_m v_m, \quad v_0 = u_0,\ v_m = w_0. \tag{3.2}$$

Remove the cycle C_0 and the internal part P° of the path P, we obtain a cycle-tree

$$T_1 = T \smallsetminus (C_0 \cup P^\circ), \tag{3.3}$$

which has one fewer block cycle than T. Choose an edge z_1 on C_1 incident with w_0 and switch the sign of z_1, we obtain a cycle-tree T_1'. If T is Eulerian, so is T_1', for the block cycle C_1 has one fewer intersection vertex in T_1' than in T. This procedure will be recalled in the proof of Lemma 3.1 and Theorem 3.2.

Lemma 3.1. *Let T be a cycle-tree. Then there exists a closed walk W on T that uses each edge of block cycles once and each edge of block paths twice, and crosses from one block to the other block at each cut-vertex.*

Moreover, each such W is a closed walk of minimum length that uses all edges of T, and vice versa.

Proof. If T has only one block cycle, then T is the cycle C_0 and can be written as a closed walk in (3.1). If T has two or more block cycles, then by induction there is a closed walk W_1 on T_1 in (3.3) such that W_1 crosses from one block to the other block at each intersection vertex. Then $W = C_0PW_1P^{-1}$ is the required closed walk on T; see Figure 8. The minimality of length is trivial. The part of vice versa is also trivial. □

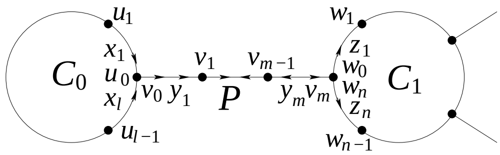

FIGURE 8. End-block cycle of a cycle-tree T.

Theorem 3.2 (Existence and Uniqueness of Direction on Eulerian Cycle-Tree)**.** *Let T be a cycle-tree. Then T satisfies the Parity Condition if and only if there exists a (unique) direction ε_T on T (up to opposite sign).*

Proof. "⇒": We proceed by induction on the number of block cycles of T. When T has only one block cycle, then T is a cycle itself, and the cycle has to be balanced. It is clear that a balanced cycle has a unique direction up to opposite sign.

Assume that T has two or more block cycles. Then T_1 in (3.3) is a cycle-tree having one fewer block cycle than T. Switch the sign of the edge z_1 in T_1, we obtain an Eulerian cycle-tree T_1'. By induction there exists a unique direction $\varepsilon_{T_1'}$ (up to opposite sign) on T_1'. Let us switch the sign of z_1 in T_1' back to the sign of z_1 in T_1 and define an orientation ε_{T_1} on T_1 by setting $\varepsilon_{T_1} = \varepsilon_{T_1'}$ for all vertex-edge pairs except

$$\varepsilon_{T_1}(w_0, z_1) = -\varepsilon_{T_1'}(w_0, z_1).$$

Then (C_1, ε_{T_1}) has either a sink or a source at w_0. Let ε_P be a direction on P such that $\varepsilon_P(v_m, y_m) = -\varepsilon_{T_1}(w_0, z_1)$, and ε_{W_0} be a direction on W_0 such that $\varepsilon_{W_0}(u_l, x_l) = -\varepsilon_P(v_0, y_1)$. Then the joint orientation $\varepsilon_{W_0} \vee \varepsilon_P \vee \varepsilon_{T_1}$ gives rise to a direction ε_T on T; see Figure 8.

Let ε_T' be an arbitrary direction on T. Then ε_T' induces directions $\varepsilon_{W_0}', \varepsilon_P', \varepsilon_{T_1'}'$ on W_0, P, T_1' respectively, where

$$\begin{aligned} \varepsilon_{T_1'}'(w_0, z_1) &= -\varepsilon_T'(w_0, z_1), \\ \varepsilon_P'(v_m, y_m) &= -\varepsilon_T(w_0, z_n), \\ \varepsilon_{W_0}'(u_l, x_l) &= -\varepsilon_P'(v_0, y_1). \end{aligned}$$

Then by induction we have that $\varepsilon_{T_1'}' = \pm\varepsilon_{T_1'}$. It follows that $\varepsilon_P' = \pm\varepsilon_P$ and $\varepsilon_{W_0}' = \pm\varepsilon_{W_0}$. Hence $\varepsilon_T' = \pm\varepsilon_T$; see Figure 8. This shows that the direction ε_T is unique up to opposite sign.

"⇐": Again, we proceed by induction on the number of block cycles. Given a direction ε_T of T. By Lemma 3.1 there exists a closed walk W on T that uses each edge of block cycles once and each edge of block paths twice. Then (W, ε_T) is a directed Eulerian walk by definition of direction on T. If T has only one block cycle C, then $T = C$ and it is trivially true, for the cycle C has zero number of intersection vertices and is balanced by (1.8).

Assume that T has two or more block cycles. Let (W_1, ε_{W_1}) be the restriction of (W, ε_W) to T_1 in (3.3). Then (W_1, ε_{W_1}) is a directed walk, having either a sink or a source at w_0. Switch the sign of the edge z_1 in T and its orientation at w_0. We obtain a directed Eulerian walk $(W_1', \varepsilon_{W_1}')$ on T_1' that uses each edge of block cycles once and each edge of block paths twice. By induction all block cycles of T_1' satisfy the Parity Condition. Thus all block cycles of T_1 other than C_1 satisfies the Parity Condition. Let us switch the sign of z_1 in T_1' back to the sign of z_1 in T. Since C_1 has one fewer intersection vertex in T_1' than that in T, we see that C_1 satisfies the Parity Condition in T. Since (C_0, ε_T) has either a sink or a source at u_0, it forces that C_0 is unbalanced. Hence C_0 also satisfies the Parity Condition. □

Lemma 3.3. *Let W be a minimal Eulerian walk with a direction ε_W. Then*

(a) (W, ε_W) *is midway-back avoided.*

(b) *Each double vertex in W is a cut-point of $\Sigma(W)$.*

Proof. (a) Let W be written as

$$W = u_0x_1u_1 \dots u_{\alpha-1}x_\alpha u_\alpha \dots u_{\beta-1}x_\beta u_\beta \dots u_{\ell-1}x_\ell u_\ell, \quad u_\ell = u_0,$$

with $u_\alpha = u_\beta$, where $0 < \alpha < \beta < \ell$. If $\varepsilon_W(u_\beta, x_\beta) = -\varepsilon_W(u_\alpha, x_{\alpha+1})$, then (W', ε_W) is a directed Eulerian walk, where

$$W' = u_\alpha x_{\alpha+1}u_{\alpha+1} \dots u_{\beta-1}x_\beta u_\beta,$$

and W' is properly contained in W as multisets, which contradicts the minimality of (W, ε_W). Then we must have $\varepsilon_W(u_\beta, x_\beta) = \varepsilon_W(u_\alpha, x_{\alpha+1})$. This means that (W, ε_W) is midway-back avoided.

(b) The proof is similar to that of Lemma 2.4(b). Note that W has only possible double vertices and possible double edges. Let $u = u_\delta = u_\eta$ be a double vertex of W with indices $\delta < \eta$. Remove u from $\Sigma(W)$. Then $\Sigma(W) \smallsetminus \{u\}$ is the union of two open walks

$$x_{\delta+1}u_{\delta+1}x_{\delta+2} \cdots x_{\eta-1}u_{\eta-1}x_\eta, \quad x_{\eta+1}u_{\eta+1}x_{\eta+2} \dots x_{\delta-1}u_{\delta-1}x_\delta.$$

Suppose u is not a cut-point of $\Sigma(W)$. Then the two open walks must intersect, say, at the vertex $u_\mu = u_\nu$ with indices $\delta < \mu < \eta < \nu$; see Figure 7. Since (W, ε_W) is midway-back avoided, we have

$$\varepsilon_W(u_\eta, x_\eta) = \varepsilon_W(u_\delta, x_{\delta+1}), \quad \varepsilon_W(u_\nu, x_\nu) = \varepsilon_W(u_\mu, x_{\mu+1}).$$

Since (W, ε_W) is directed, we futher have

$$\varepsilon_W(u_\delta, x_\delta) = \varepsilon_W(u_\eta, x_{\eta+1}), \quad \varepsilon_W(u_\mu, x_\mu) = \varepsilon_W(u_\nu, x_{\nu+1}).$$

We see that (W_1, ε_W) and (W_2, ε_W), where

$$W_1 = u_\delta x_{\delta+1}u_{\delta+1} \dots u_{\mu-1}x_\mu u_\mu(u_\nu)x_\nu u_{\nu-1} \dots u_{\eta+1}x_{\eta+1}u_\eta,$$

$$W_2 = u_\eta x_\eta u_{\eta-1} \dots u_{\mu+1}x_{\mu+1}u_\mu(u_\nu)x_{\nu+1}u_{\nu+1} \dots u_{\delta-1}x_\delta u_\delta,$$

are directed closed positive walks, and are contained properly in (W, ε_W) as multisets. This is contradictory to the minimality of (W, ε_W). □

Theorem 3.4 (Characterization of Minimal Eulerian Walk)**.** *Let W be a minimal Eulerian walk with a direction ε_W. Then $\Sigma(W)$ is an Eulerian cycle-tree T, W uses each edge of block cycles once and each edge of block paths twice of T, and ε_W induces a direction ε_T on T. Moreover,*

$$f_{(W,\varepsilon_W)} = [\varepsilon, \varepsilon_T] \cdot I_T. \tag{3.4}$$

Proof. It follows from Lemma 3.3 and Theorem 2.5 that $\Sigma(W)$ is an Eulerian cycle-tree. Since $\varepsilon_W(u,x) = \varepsilon_T(u,x)$ for vertex-edge pairs (u,x) on T, the identity (3.4) follows immediately from definitions of $f_{(W,\varepsilon_W)}$ by (1.10) and I_T by (1.12). $\square$

Theorem 3.5 (Minimality of Eulerian Cycle-Tree)**.** *Let T be an Eulerian cycle-tree with a direction ε_T. Then T is minimal in the sense that if T_1 is an Eulerian cycle-tree contained in T and block paths of T_1 are block paths of T then $T_1 = T$.*

Moreover, if W is a closed walk on T that uses each edge of block cycles once and each edge of block paths twice, then (W, ε_T) is a minimal directed Eulerian walk.

In particular, each directed closed positive walk found by FRA is a minimal directed Eulerian walk.

Proof. Suppose there is an Eulerian cycle-tree T_1 contained properly in T, such that all block paths of T_1 are block paths of T. Then there exists an edge $e \in E(T) \smallsetminus E(T_1)$, having an end-vertex v on a block cycle C of T_1. The vertex v must be an intersection vertex in T but not an intersection vertex in T_1. Let e_1, e_2 be two edges (could be an identical loop) on C, having a common end-vertex v. Then $\varepsilon_T(v, e_1) = -\varepsilon_T(v, e_2)$ in T_1 and $\varepsilon_T(v, e_1) = \varepsilon_T(v, e_2)$ in T. This is a contradiction.

Let W be a required closed walk on T. Then (W, ε_T) is a directed Eulerian walk by Lemma 3.1 and by definition of ε_T. Let W_1 be a minimal Eulerian walk on T, contained in W as multisets. Then $T_1 = \Sigma(W_1)$ is contained in T and is an Eulerian cycle-tree by Theorem 3.4. Clearly, all block cycles of T_1 are block cycles of T. Since edges of block paths of T are double edges in W, edges of block paths of T_1 are double edges in W_1, and double edges in W_1 are double edges in W, it follows that all block paths of T_1 are block paths of T. Thus $T_1 = T$ by the first part of the theorem. Therefore $M(W_1) = M(W)$; this means that W is a minimal Eulerian walk on T.

The last part follows from Corollary 2.6. $\square$

Corollary 3.6 (Characterization and Classification of Circuits)**.** *Let (W, ε_W) be a minimal directed Eulerian walk. Then the following statements are equivalent.*

(a) (W, ε_W) *is elementary.*
(b) $f_{(W,\, \varepsilon_W)}$ *is elementary.*
(c) $\Sigma(W)$ *is a circuit.*

Moreover, circuits are classified into Types I, II, III.

Proof. (a) $\Rightarrow$ (b): Suppose $f_{(W,\, \varepsilon_W)}$ is not elementary, that is, there is a flow g such that $\operatorname{supp} g \subsetneq \operatorname{supp} f_{(W,\, \varepsilon_W)}$. We may require $\Sigma(\operatorname{supp} g)$ to be connected. By Lemma 2.1 there exists a directed closed positive walk (W_1, ε_g) on $\Sigma(\operatorname{supp} g)$ such that $g = f_{(W_1,\, \varepsilon_g)}$. Since $\operatorname{supp} W_1 = \operatorname{supp} g$ and $\operatorname{supp} f_{(W,\, \varepsilon_W)} = \operatorname{supp} W$, then $\operatorname{supp} W_1 \subsetneq \operatorname{supp} W$. Thus (W, ε_W) is not elementary by definition, which is contradictory to (a).

(a) $\Leftarrow$ (b): Suppose (W, ε_W) is not elementary, that is, there exists a minimal directed Eulerian walk (W_1, ε_{W_1}) such that $\operatorname{supp} W_1 \subsetneq \operatorname{supp} W$. Since $\operatorname{supp} W_1 = \operatorname{supp} f_{(W_1,\, \varepsilon_{W_1})}$ and $\operatorname{supp} W = \operatorname{supp} f_{(W,\, \varepsilon_W)}$, then $\operatorname{supp} f_{(W_1, \varepsilon_{W_1})} \subsetneq \operatorname{supp} f_{(W, \varepsilon_W)}$. Thus $f_{(W,\, \varepsilon_W)}$ is not elementary by definition, which is contradictory to (b).

(a) $\Rightarrow$ (c): Suppose $\Sigma(W)$ is not a circuit, that is, there exists an Eulerian cycle-tree T_1 contained properly in $\Sigma(W)$. Let ε_{T_1} be a direction on T_1, and W_1 a closed walk that uses each edge of block cycles once and each edge of block paths twice of T_1. Then (W_1, ε_{T_1}) is a minimal directed Eulerian walk by Theorem 3.5

and $\operatorname{supp} W_1 \subsetneq \operatorname{supp} W$. Thus (W, ε_W) is not elementary by definition, which is contradictory to (c).

(a) $\Leftarrow$ (c): Suppose (W, ε_W) is not elementary, that is, there exists a minimal directed Eulerian walk (W_1, ε_{W_1}) such that $\operatorname{supp} W_1 \subsetneq \operatorname{supp} W$. Then $\Sigma(W_1)$ is an Eulerian cycle-tree by Theorem 3.4 and is properly contained in $\Sigma(W)$. Thus $\Sigma(W)$ is not a circuit by deinition, which is contradictory to (a).

Now an Eulerian cycle-tree T be further a circuit. Then T contains at most one block path (of possible zero length). Otherwise, suppose there are two or more block paths in T, then one block path together with its two block cycles form an Eulerian cycle-tree, which is properly contained in T; this means that T is not a circuit, a contradiction. If there is no block path in T, then T must be a single balanced cycle, which is a circuit of Type I.

If T contains exactly one block path, the length of the block path is either zero or positive. In the case of zero length for the block path, T consists of two block cycles having a common vertex, which is a circuit of Type II. In the case of positive length for the block path, T consists of two block cycles and the block path connecting them, which is a circuit of Type III. □

Theorem 3.7 (Half-Integer Scale Decomposition). *Let T be an Eulerian cycle-tree with a direction ε_T. Let W be a closed walk on T that uses each edge of block cycles once and each edge of block paths twice. If T is not a circuit, then W can be divided into the form*

$$W = C_0P_1C_1P_2 \cdots P_kC_kP_{k+1}, \quad k \geq 1, \tag{3.5}$$

satisfying the following four conditions:

(i) *$\{C_i\}$ is the collection of all end-block cycles of T and P_i are simple open paths of positive lengths.*

(ii) *Each edge of non-end-block cycles appears in exactly one of the paths P_i, and each edge of block paths appears in exactly two of the paths P_i.*

(iii) *Each $(C_iP_{i+1}C_{i+1}, \varepsilon_T)$ $(0 \leq i \leq k)$ is a directed circuit of Type III with $C_{k+1} = C_0$.*

(iv) *Half-integer scale decomposition*

$$I_T = \frac{1}{2}\sum_{i=0}^{k} I_{\Sigma(C_iP_{i+1}C_{i+1})}. \tag{3.6}$$

Proof. We proceed by induction on the number of block paths of T, including those of zero length. If T does not contain block path, then T is a circuit of Types I. If T contains exactly one block path of zero length, then T is a circuit of Type II. If T contains exactly one block path of positive length, then T is a circuit of Type III.

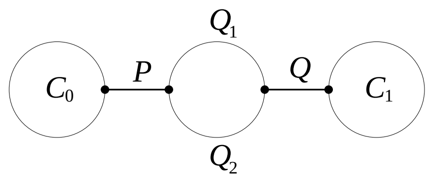

FIGURE 9. An Eulerian cycle-tree with two block paths.

When T has exactly two block paths (of possible zero length), then T has the form in Figure 9. Since W crosses each cut-vertex from one block to the other block, then W can be written as $W = C_0P_1C_1P_2$, where $P_1 = PQ_1Q$, $P_2 = Q^{-1}Q_2^{-1}P^{-1}$. Then $C_0P_1C_1, C_1P_2C_0$ are circuits of Type III. We thus have the decomposition

$$I_T = \frac{1}{2}I_{\Sigma(C_0P_1C_1)} + \frac{1}{2}I_{\Sigma(C_1P_2C_0)}.$$

When T has three or more block paths (of possible zero length), choose an end-block cycle C and a block path P (of possible zero length) having its initial vertex u on C and its terminal vertex v on another block cycle C'. Since T has at least three block paths, the cycle C' cannot be a loop; so all edges of C' are not loops. Choose an edge x on C' at v, change the sign of x, and remove the cycle C and the internal part of P from T. We obtain an Eulerian cycle-tree T'; see Figures 10 and 11. Then W can be written as $W = CPW'P^{-1}$, where W' is a closed walk on T' that uses each edge of block cycles once and each edge of block paths twice. Thus $(W', \varepsilon_{T'})$ is a minimal Eulerian walk, where $\varepsilon_{T'}$ is a direction of T' and $\varepsilon_{T'} = \varepsilon_T$ except $\varepsilon_{T'}(v,x) = -\varepsilon_T(v,x)$. By induction W' can be written as

$$W' = C_0'P_1'C_1'P_2' \cdots P_k'C_k'P_{k+1}',$$

satisfying the conditions (i)–(iv). There are two cases: C' is either an end-block cycle of T', or C' is not an end-block cycle of T'.

In the case that C' is an end-block cycle of T', we may assume $C_k' = C'$, having its unique intersection vertex at w in T'. Let us write C' as a closed path $C_k' = P'Q'$, where P' is a path from v to w on C' and Q' is the other path from w to v on C'. Note that P_k' is a path whose terminal vertex is w, and P_{k+1}' is a path whose initial vertex is w; see Figure 10. Set $C_i = C_i'$ $(0 \le i \le k-1)$, $P_i = P_i'$ $(1 \le i \le k-1)$,

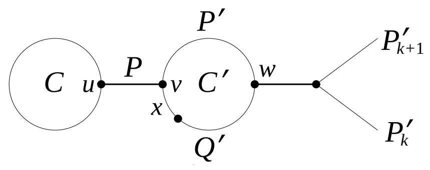

FIGURE 10. C' has two intersection vertices.

and

$$P_k = P_k'Q'P^{-1}, \quad C_k = C, \quad P_{k+1} = PP'P_{k+1}'.$$

Then $W = C_0P_1C_1P_2 \cdots P_kC_kP_{k+1}$ is a closed walk on T with direction ε_T, satisfying the conditions (i)–(iv).

In the case that C' is not an end-block cycle of T', we may assume that P_{k+1}' contains the vertex v and the edge x. Let us write $P_{k+1}' = P'Q'$, where P' is a path whose terminal vertex is v and Q' is a path whose initial vertex is v; see Figure 11. Set $C_i = C_i'$ $(0 \le i \le k)$, $P_i = P_i'$ $(1 \le i \le k)$, and

$$P_{k+1} = P'P^{-1}, \quad C_{k+1} = C, \quad P_{k+2} = PQ'.$$

Then $W = C_0P_1C_1P_2 \cdots P_{k+1}C_{k+1}P_{k+2}$ is a closed walk on T with direction ε_T, satisfying the conditions (i)–(iv). $\square$

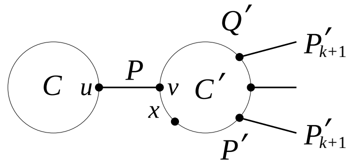

FIGURE 11. C' has more than two intersection vertices.

Problem. An Eulerian cycle-tree is said to be *bridgeless* if it does not contain block paths of positive length. The indicator function of a bridgeless Eulerian cycle-tree has constant value 1 on its support. It should be interesting to consider integral flows f such that $\Sigma(f)$ is connected and has no bridges; we will call such integral flows as *bridgeless flows*. A bridgeless flow f is said to be *bridgeless decomposable* if there exist nonzero bridgeless flows f_1, f_2 such that $f = f_1 + f_2$, where f_i have the same sign, that is, $f_1 \cdot f_2 \geq 0$. It would be interesting to classify *bridgeless indecomposable flows*, that is, the integral flows that are not bridgeless decomposable.

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